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Accurate equations for laminated composite deep thick shells

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Abstract

This paper derives accurate equations of elastic deformation for laminated composite deep, thick shells. The equations include shells with a pre-twist and accurate force and moment resultants which are considerably different than those used for plates. This is due to the fact that the stresses over the thickness of the shell have to be integrated on a trapezoidal-like cross-section of a shell element to obtain the stress resultants. Numerical results are obtained and showed that accurate stress resultants are needed for laminated composite deep thick shells, especially if the curvature is not spherical. A consistent set of equations of motion, energy functionals and boundary conditions are also derived. These may be used in obtaining exact solutions or approximate ones like the Ritz or finite element methods. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Love's thin shell theory (Love, 1892) is among the first developments of the theory of shells. In this theory, Love introduced his first approximation for bending analysis of shells. This approximation defines a linear analysis of thin shells, in which various assumptions were introduced. Among these assumptions, strains and displacements are small such that second- and higher-order terms can be neglected. Also, Love assumed the thickness of the shell to be small compared with other shell parameters, the transverse stress to be small compared with other stresses in shells and normals to the undeformed surface to remain straight and normal to the deformed surface. The first of these assumptions defines a linear analysis of shells. This assumption needs to be relaxed if the strains and/or displacements become large. Displacement is considered definitely large if it exceeds the thickness of the shell. This is typical for thin shells. Nonlinear behavior can be observed even before this for various boundary conditions. A recent study (Qatu, 1994) concluded that this assumption generally applies to most of the analyses of thick shells. This is because stresses exceed allowable values before the deflection becomes large enough for the nonlinear terms to be important.

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This paper is concerned with thick, deep shells, where the remaining assumptions in Love's first approximation need to be re-examined. The thickness is no longer small compared with other shell parameters, nor do the normals of the undeformed surface remain as such. Various studies (Koiter, 1969; Gol'denveizer, 1968; Noor, 1990) concluded that even for thicker shells the transverse normal stress remains small compared with other stresses in the shell.

Since the first Love shell theory and other theories were introduced, inconsistencies appeared in many of these theories. Kadi (1973) and Leissa (1973) showed that the strain–displacement relations used by Naghdi and berry (1964) are inconsistent with regard to rigid body motion. Other theories including Love (1892) and Timoshenko and Woinowsky-Krieger (1959), although free from rigid body motion inconsistencies, introduced unsymmetric differential operators, which contradicts the theorem of reciprocity and yields imaginary numbers for natural frequencies in a free vibration analysis. Other inconsistencies appeared when the assumption of small thickness (h/R and $z/R \ll 1$) are imposed and symmetric stress resultants (i.e., $N_{\alpha\beta} = N_{\beta\alpha}$ and $M_{\alpha\beta} = M_{\beta\alpha}, \ldots$) are obtained, which is not true unless the shell is spherical. To overcome some or all the above inconsistencies, various theories were introduced including that of Sanders (1959). Earlier theories (Vlasov, 1949) tried to resolve some of these inconsistencies by expanding the terms z/R that appear in the denominator of the stress resultant equations using a Taylor series.

On the other hand, it was noted as early as 1877 that rotary inertia terms are important in the analysis of vibrating systems (Rayleigh, 1877). More than forty years later, Timoshenko (1921) showed that shear deformation terms are at least as important. Since then researchers realized that for thick beams, plates or shells, both rotary inertia and shear deformation have to be included in any reliable theory for such components. This, when generalized, led to a necessary relaxation of some of the assumptions in Love's first approximation and shear deformation shell theories were born. Among the first of such theories were those of Vlasov (1949), Reissner (1952), Naghdi and Cooper (1956) and others. Interestingly enough, many of the theories that included shear deformation ignored the $(1+z/R)$ terms in the stress resultant equations. On the other hand, theories that included such terms truncated them later (Vlasov, 1949).

Among the most notable researchers in the area of laminated composite shells is Ambartsumian whose work is summarized in a monograph of his own (Ambartsumian, 1961). He introduced various laminated composite shell theories by expanding the stress resultant equation of earlier theories to those for anisotropic shells. Since then various survey articles appeared (Ambartsumian, 1962; Bert and Egle, 1969; Noor, 1990; Liew et al., 1997) which reviewed theories and analysis of laminated composite shells. In most of these theories, shear deformation was included and it was found that shear deformation effects for laminated composite materials are generally more important than those for isotropic materials. Unfortunately, while shear deformation shell theories, including higher-order ones (e.g. Lim and Liew, 1995; Liew and Lim, 1996), include shear deformation and rotary inertia, they fail to consider the $1+z/R$ terms in the stress resultant equations. This leads to numerous errors in the constitutive equations used for laminated deep thick shells. This was initially observed by Bert (1967) and only recently by Chang (1993) and Leissa and Chang (1996). Chang did consider this term, but truncated it using a geometric series expansion. He showed that this with only First-order Shear Deformation (FSD) theories gave more accurate results than higher-order theories in which the term was neglected (Reddy, 1984; Reddy and Liu, 1985; Librescu et al., 1989a, b).

The purpose of this work is to introduce a consistent and accurate set of equations for laminated composite thick and deep shells. The equations include accurate force and moment resultant

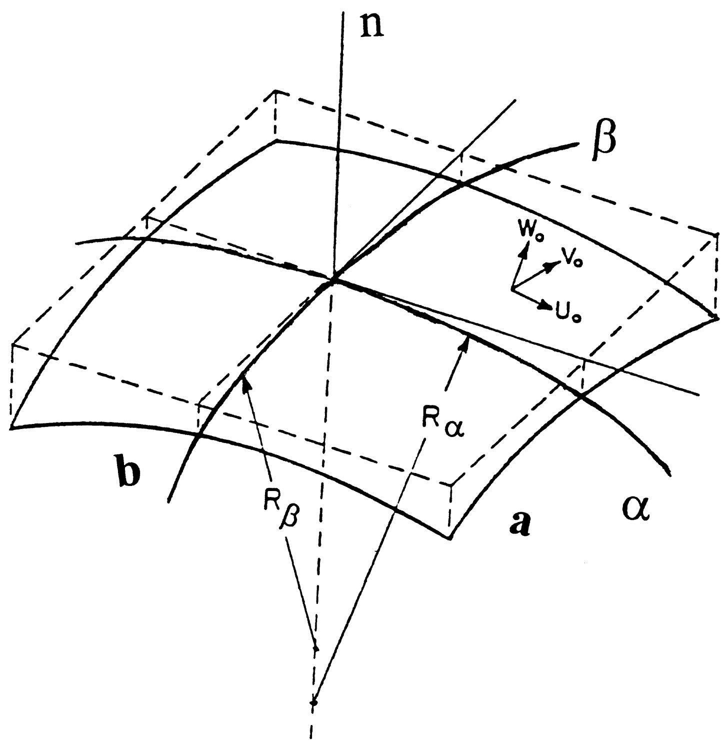

Fig. 1. A shell element in curvilinear coordinates.

equations for laminated composite deep thick shells where the $1+z/R$ terms are included in the stress resultant equations and integrated exactly. Also it will be shown that these terms can be as important as shear deformation effects in many shell configurations. The resulting force and moment resultant equations can be used in many shell theories. The equations of motion, boundary conditions are derived from the energy functionals using Hamilton's principle.

2. Kinematic relations

Consider a shell element of radii of curvature R_α and R_β and a radius of twist $R_{\alpha\beta}$ and a thickness h (Fig. 1). The length of an infinitesimal element of thickness dz located at distance z from the shell midsurface is

$$\mathrm{d}s_\alpha^{(z)} = \frac{A\,\mathrm{d}\alpha}{R_\alpha}(R_\alpha+z) = A(1+z/R_\alpha)\,\mathrm{d}\alpha$$

$$ds_\beta^{(z)} = \frac{B\,\mathrm{d}\beta}{R_\beta}(R_\beta+z) = B(1+z/R_\beta)\,\mathrm{d}\beta \tag{1}$$

where A and B are Lamé parameters (Leissa, 1973). The displacement vector $\mathbf{U}$ of an arbitrary point within the shell may be written as:

$$\mathbf{U} = u\mathbf{i}_\alpha + v\mathbf{i}_\beta + w\mathbf{i}_z \tag{2}$$

where $\mathbf{i}_\alpha$, $\mathbf{i}_\beta$ and $\mathbf{i}_z$ are unit vectors in the α-, β- and z-directions, respectively.

The strain displacement relations can be derived from the above equations (Gol'denveizer, 1953; Chang, 1993; Leissa and Chang, 1996) as

$$\begin{aligned}
\varepsilon_\alpha &= \frac{1}{(1+z/R_\alpha)}\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{v}{AB}\frac{\partial A}{\partial \beta} + \frac{w}{R_\alpha}\right) \\
\varepsilon_\beta &= \frac{1}{(1+z/R_\beta)}\left(\frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{u}{AB}\frac{\partial B}{\partial \alpha} + \frac{w}{R_\beta}\right) \\
\varepsilon_{\alpha\beta} &= \frac{1}{(1+z/R_\alpha)}\left(\frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{u}{AB}\frac{\partial A}{\partial \beta} + \frac{w}{R_{\alpha\beta}}\right) \\
\varepsilon_{\beta\alpha} &= \frac{1}{(1+z/R_\beta)}\left(\frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{v}{AB}\frac{\partial B}{\partial \alpha} + \frac{w}{R_{\alpha\beta}}\right) \\
\gamma_{\alpha z} &= \frac{1}{A(1+z/R_\alpha)}\frac{\partial w}{\partial \alpha} + A(1+z/R_\alpha)\frac{\partial}{\partial z}\left(\frac{u}{A(1+z/R_\alpha)}\right) - \frac{v}{R_{\alpha\beta}(1+z/R_\alpha)} \\
\gamma_{\beta z} &= \frac{1}{B(1+z/R_\beta)}\frac{\partial w}{\partial \beta} + B(1+z/R_\beta)\frac{\partial}{\partial z}\left(\frac{v}{B(1+z/R_\beta)}\right) - \frac{u}{R_{\alpha\beta}(1+z/R_\beta)}
\end{aligned} \tag{3}$$

Assuming that normals to the midsurface strains remain straight during deformation but not normal, the displacements can be written as

$$\begin{aligned}
u(\alpha,\beta,z) &= u_0(\alpha,\beta) + z\psi_\alpha(\alpha,B) \\
v(\alpha,\beta,z) &= v_0(\alpha,\beta) + z\psi_\beta(\alpha,B) \\
w(\alpha,\beta,z) &= w_0(\alpha,\beta)
\end{aligned} \tag{4}$$

where u_0, v_0 and w_0 are midsurface displacements of the shell and ψ_α and ψ_β are midsurface rotations. The above equations describe a typical first order shear deformation shell theory and will constitute the only assumption made in this analysis when compared with the three-dimensional theory of elasticity. Substituting eqns (4) into (3) yields:

$$\begin{aligned}
\varepsilon_\alpha &= \frac{1}{(1+z/R_\alpha)}(\varepsilon_{0\alpha} + z\kappa_\alpha) \\
\varepsilon_\beta &= \frac{1}{(1+z/R_\beta)}(\varepsilon_{0\beta} + z\kappa_\beta) \\
\varepsilon_{\alpha\beta} &= \frac{1}{(1+z/R_\alpha)}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta})
\end{aligned}$$

$$\varepsilon_{\beta\alpha} = \frac{1}{(1+z/R_\beta)}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha})$$

$$\gamma_{\alpha z} = \frac{1}{(1+z/R_\alpha)}(\gamma_{0\alpha z} - z(\psi_\beta/R_{\alpha\beta}))$$

$$\gamma_{\beta z} = \frac{1}{(1+z/R_\beta)}(\gamma_{0\beta z} - z(\psi_\alpha/R_{\alpha\beta})) \tag{5}$$

where the midsurface strains are:

$$\varepsilon_{0\alpha} = \frac{1}{A}\frac{\partial u_0}{\partial\alpha} + \frac{v_0}{AB}\frac{\partial A}{\partial\beta} + \frac{w_0}{R_\alpha}$$

$$\varepsilon_{0\beta} = \frac{1}{B}\frac{\partial v_0}{\partial\beta} + \frac{u_0}{AB}\frac{\partial B}{\partial\alpha} + \frac{w_0}{R_\beta}$$

$$\varepsilon_{0\alpha\beta} = \frac{1}{A}\frac{\partial v_0}{\partial\alpha} - \frac{u_0}{AB}\frac{\partial A}{\partial\beta} + \frac{w_0}{R_{\alpha\beta}}$$

$$\varepsilon_{0\beta\alpha} = \frac{1}{B}\frac{\partial u_0}{\partial\beta} - \frac{v_0}{AB}\frac{\partial B}{\partial\alpha} + \frac{w_0}{R_{\alpha\beta}}$$

$$\gamma_{0\alpha z} = \frac{1}{A}\frac{\partial w_0}{\partial\alpha} - \frac{u_0}{R_\alpha} - \frac{v_0}{R_{\alpha\beta}} + \psi_\alpha$$

$$\gamma_{0\beta z} = \frac{1}{B}\frac{\partial w_0}{\partial\beta} - \frac{v_0}{R_\beta} - \frac{u_0}{R_{\alpha\beta}} + \psi_\beta \tag{6}$$

and the curvature and twist changes are:

$$\kappa_\alpha = \frac{1}{A}\frac{\partial\psi_\alpha}{\partial\alpha} + \frac{\psi_\beta}{AB}\frac{\partial A}{\partial\beta}, \quad \kappa_\beta = \frac{1}{B}\frac{\partial\psi_\beta}{\partial\beta} + \frac{\psi_\alpha}{AB}\frac{\partial B}{\partial\alpha}$$

$$\kappa_{\alpha\beta} = \frac{1}{A}\frac{\partial\psi_\beta}{\partial\beta} - \frac{\psi_\alpha}{AB}\frac{\partial A}{\partial\beta}, \quad \kappa_{\beta\alpha} = \frac{1}{B}\frac{\partial\psi_\alpha}{\partial\beta} - \frac{\psi_\beta}{AB}\frac{\partial B}{\partial\alpha} \tag{7}$$

3. Force and moment resultants

The stress–strain relationship for a typical nth lamina (typically called monoclinic) in a laminated composite shell made of N laminae (Fig. 5) is (Whitney, 1987):

$$\begin{bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_z \\ \sigma_{\beta z} \\ \sigma_{\alpha z} \\ \sigma_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} & 0 & 0 & \bar{Q}_{26} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} & 0 & 0 & \bar{Q}_{36} \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{36} & 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_z \\ \gamma_{\beta z} \\ \gamma_{\alpha z} \\ \gamma_{\alpha\beta} \end{bmatrix} \tag{8}$$

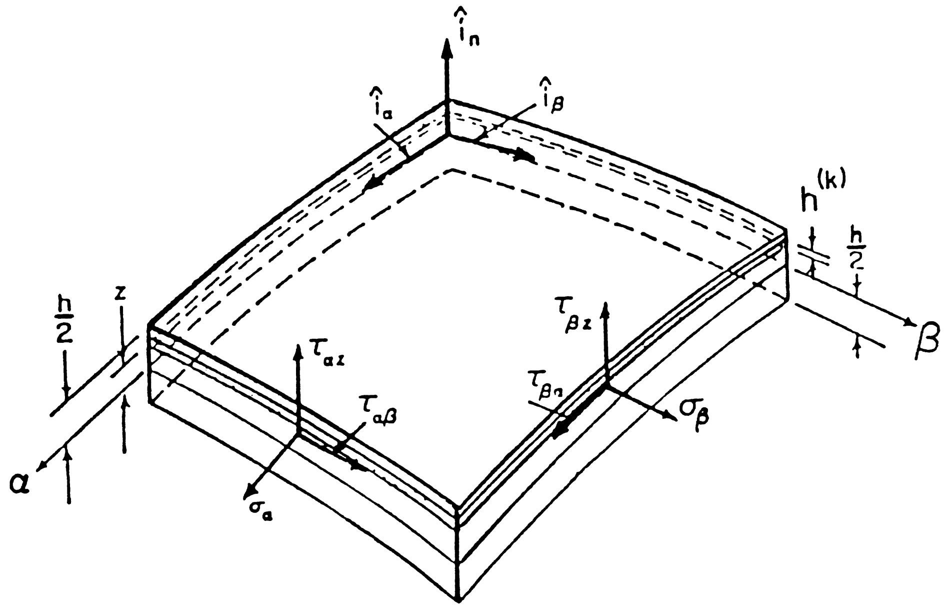

Fig. 2. Stresses in shell coordinates.

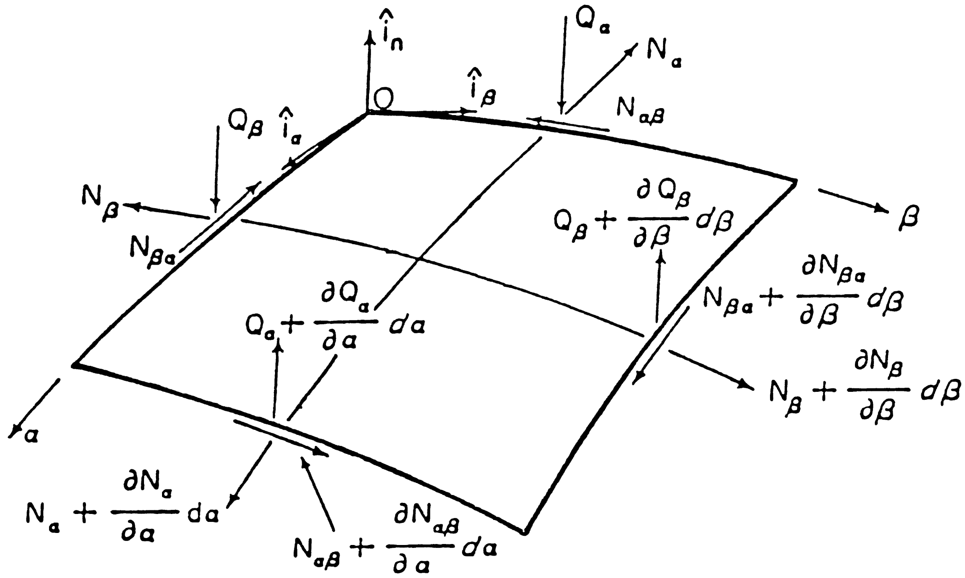

Fig. 3. Force resultants in shell coordinates.

The positive notations of the stresses are shown in Fig. 2. It should be mentioned that no stretching is assumed in the z-direction in the above equations (i.e., $\varepsilon_z = 0$). This actually is directly derived from the third equation in (4).

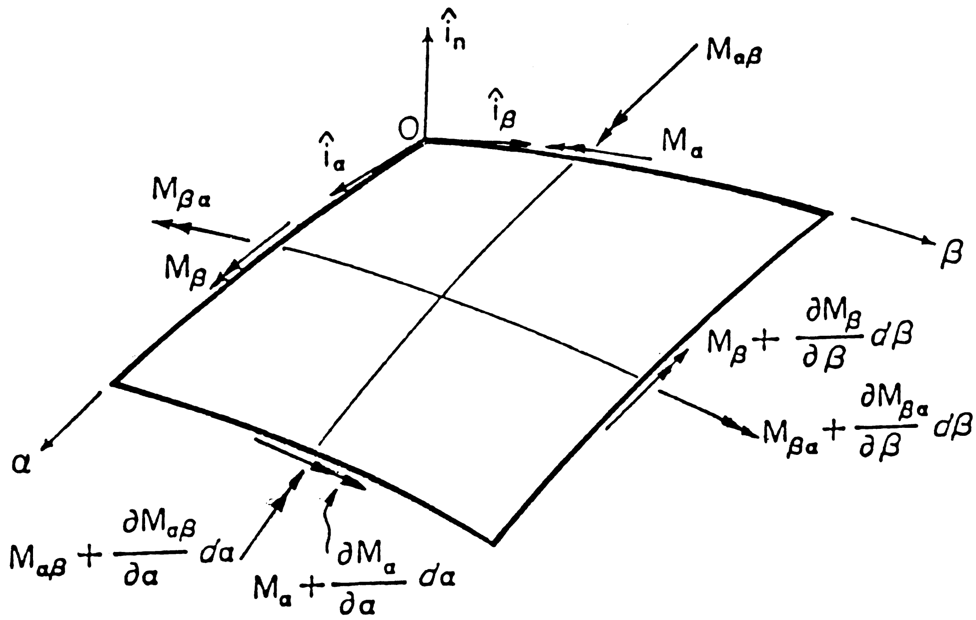

Fig. 4. Moment resultants in shell coordinates.

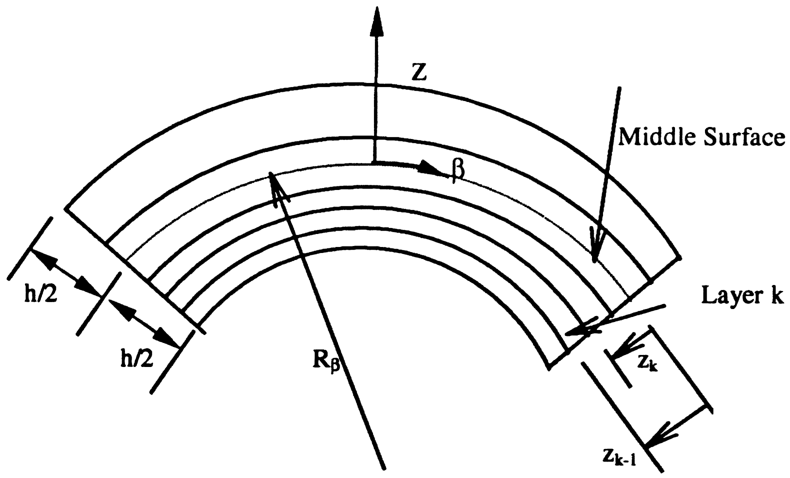

Fig. 5. Lamination parameters in shells.

The force and moment resultants (Figs 3 and 4) are obtained by integrating the stresses over the shell thickness. The normal and shear force resultants are:

$$\begin{bmatrix} N_\alpha \\ N_{\alpha\beta} \\ Q_\alpha \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_\alpha \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{bmatrix} (1 + z/R_\beta)\, \mathrm{d}z$$

$$\begin{bmatrix} N_\beta \\ N_{\beta\alpha} \\ Q_\beta \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_\beta \\ \sigma_{\alpha\beta} \\ \psi_{\beta z} \end{bmatrix} (1+z/R_\alpha)\,\mathrm{d}z \tag{9}$$

The bending and twisting moment resultants, as well as higher-order shear resultant terms are:

$$\begin{bmatrix} M_\alpha \\ M_{\alpha\beta} \\ P_\alpha \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_\alpha \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{bmatrix} (1+z/R_\beta)z\,\mathrm{d}z$$

$$\begin{bmatrix} M_\beta \\ M_{\beta\alpha} \\ P_\beta \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_\beta \\ \sigma_{\alpha\beta} \\ \sigma_{\beta z} \end{bmatrix} (1+z/R_\alpha)z\,\mathrm{d}z \tag{10}$$

where P_α and P_β are higher-order shear terms, needed only if the radius of twist curvature exist (i.e. $R_{\alpha\beta} \neq \infty$). It should be mentioned that although the stresses $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$, the stress resultants $N_{\alpha\beta} \neq N_{\beta\alpha}$ and $M_{\alpha\beta} \neq M_{\beta\alpha}$, unless $R_\alpha = R_\beta$ which is the case only for spherical shells or flat plates. Substituting eqns (5) and (8) into (9) and (10), yields the following equations for the normal, in-plane and shear forces:

$$N_\alpha = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\bar{Q}_{11}\varepsilon_\alpha + \bar{Q}_{12}\varepsilon_\beta + \bar{Q}_{16}\gamma_{\alpha\beta}](1+z/R_\beta)\,\mathrm{d}z$$

$$= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{11}(\varepsilon_{0\alpha} + z\kappa_\alpha)\left(\frac{1+z/R_\beta}{1+z/R_\alpha}\right) + \bar{Q}_{12}(\varepsilon_{0\beta} + z\kappa_\beta) + \bar{Q}_{16}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta})\left(\frac{1+z/R_\beta}{1+z/R_\alpha}\right) + \bar{Q}_{16}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha}) \right]\mathrm{d}z$$

$$N_{\alpha\beta} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{16}(\varepsilon_{0\alpha} + z\kappa_\alpha)\left(\frac{1+z/R_\beta}{1+z/R_\alpha}\right) + \bar{Q}_{26}(\varepsilon_{0\beta} + z\kappa_\beta) + \bar{Q}_{16}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta})\left(\frac{1+z/R_\beta}{1+z/R_\alpha}\right) + \bar{Q}_{66}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha}) \right]\mathrm{d}z$$

$$Q_\alpha = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{55}\left(\gamma_{0\alpha z} - z\frac{\psi_\beta}{R_{\alpha\beta}}\right)\left(\frac{1+z/R_\beta}{1+z/R_\alpha}\right) + \bar{Q}_{45}\left(\gamma_{0\beta z} - z\frac{\psi_\alpha}{R_{\alpha\beta}}\right) \right]\mathrm{d}z$$

$$N_\beta = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{12}(\varepsilon_{0\alpha} + z\kappa_\alpha) + \bar{Q}_{22}(\varepsilon_{0\beta} + z\kappa_\beta)\left(\frac{1+z/R_\alpha}{1+z/R_\beta}\right) + \bar{Q}_{26}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta}) + \bar{Q}_{26}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha})\left(\frac{1+z/R_\alpha}{1+z/R_\beta}\right) \right]\mathrm{d}z$$

$$N_{\beta\alpha} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{16}(\varepsilon_{0\alpha} + z\kappa_\alpha) + \bar{Q}_{26}(\varepsilon_{0\beta} + z\kappa_\beta) \left(\frac{1 + z/R_\alpha}{1 + z/R_\beta} \right) + \bar{Q}_{66}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta}) + \bar{Q}_{66}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha}) \left(\frac{1 + z/R_\alpha}{1 + z/R_\beta} \right) \right] \mathrm{d}z$$

$$Q_\beta = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{44} \left(\gamma_{0\beta z} - z \frac{\psi_\alpha}{R_{\alpha\beta}} \right) \left(\frac{1 + z/R_\alpha}{1 + z/R_\beta} \right) + \bar{Q}_{45} \left(\gamma_{0\alpha z} - z \frac{\psi_\beta}{R_{\alpha\beta}} \right) \right] \mathrm{d}z \tag{11a}$$

The higher-order shear terms, needed for pre-twisted shells are:

$$P_\alpha = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{55} \left(\gamma_{0\alpha z} - z \frac{\psi_\beta}{R_{\alpha\beta}} \right) \left(\frac{1 + z/R_\beta}{1 + z/R_\alpha} \right) + \bar{Q}_{45} \left(\gamma_{0\beta z} - z \frac{\psi_\alpha}{R_{\alpha\beta}} \right) \right] z \, \mathrm{d}z$$

$$P_\beta = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{44} \left(\gamma_{0\beta z} - z \frac{\psi_\alpha}{R_{\alpha\beta}} \right) \left(\frac{1 + z/R_\alpha}{1 + z/R_\beta} \right) + \bar{Q}_{45} \left(\gamma_{0\alpha z} - z \frac{\psi_\beta}{R_{\alpha\beta}} \right) \right] z \, \mathrm{d}z \tag{11b}$$

The following equations are also obtained for the bending and twisting moments:

$$M_\alpha = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\bar{Q}_{11}\varepsilon_\alpha + \bar{Q}_{12}\varepsilon_\beta + \bar{Q}_{16}\gamma_{\alpha\beta}](1 + z/R_\beta) z \, \mathrm{d}z$$

$$= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{11}(\varepsilon_{0\alpha} + z\kappa_\alpha) \left(\frac{1 + z/R_\beta}{1 + z/R_\alpha} \right) + \bar{Q}_{12}(\varepsilon_{0\beta} + z\kappa_\beta) + \bar{Q}_{16}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta}) \left(\frac{1 + z/R_\beta}{1 + z/R_\alpha} \right) + \bar{Q}_{16}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha}) \right] z \, \mathrm{d}z$$

$$M_{\alpha\beta} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{16}(\varepsilon_{0\alpha} + z\kappa_\alpha) \left(\frac{1 + z/R_\beta}{1 + z/R_\alpha} \right) + \bar{Q}_{26}(\varepsilon_{0\beta} + z\kappa_\beta) + \bar{Q}_{16}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta}) \left(\frac{1 + z/R_\beta}{1 + z/R_\alpha} \right) + \bar{Q}_{66}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha}) \right] z \, \mathrm{d}z \tag{12a}$$

$$M_\beta = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{12}(\varepsilon_{0\alpha} + z\kappa_\alpha) + \bar{Q}_{22}(\varepsilon_{0\beta} + z\kappa_\beta) \left(\frac{1 + z/R_\alpha}{1 + z/R_\beta} \right) + \bar{Q}_{26}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta}) + \bar{Q}_{26}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha}) \left(\frac{1 + z/R_\alpha}{1 + z/R_\beta} \right) \right] z \, \mathrm{d}z$$

$$M_{\beta\alpha} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \left[\bar{Q}_{16}(\varepsilon_{0\alpha} + z\kappa_\alpha) + \bar{Q}_{26}(\varepsilon_{0\beta} + z\kappa_\beta) \left(\frac{1 + z/R_\alpha}{1 + z/R_\beta} \right) + \bar{Q}_{66}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta}) + \bar{Q}_{66}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha}) \left(\frac{1 + z/R_\alpha}{1 + z/R_\beta} \right) \right] z \, \mathrm{d}z \tag{12b}$$

As can be seen in the above equations, the term $(1+z/R_n)$ where n is either α or β, in the denominator creates difficulties in carrying out the integration. Such difficulties do not exist for plates. In most thin shell theories, the term was ignored (Leissa, 1973). In some thin shell theories (e.g., Vlasov's), the term was expanded in a geometric series form. Numerical investigations revealed that for thin shells, such expansion did not introduce better results. This can be understandable for thin shells where the term $(1+z/R_n)$ is between 0.98 ad 1.02 depending on the value of z.

For thick shells, shear deformation should be considered. However, many researchers fail to include the above term in the stress resultant equations (Reddy, 1984; Reddy and Liu, 1985). When the term was expanded and then truncated in a recent presentation (Chang, 1993; Leissa and Chang, 1996), better results were obtained when compared with three-dimensional recent results (Bhimaraddi, 1991). The Appendix shows the derivation in which the term is expanded in a geometric series and then truncated. In this paper, exact stress resultants are obtained when the term is included and the integration is carried out exactly. This will yield the following stress resultant equations:

$$\begin{bmatrix} N_\alpha \\ N_\beta \\ N_{\alpha\beta} \\ N_{\beta\alpha} \\ M_\alpha \\ M_\beta \\ M_{\alpha\beta} \\ M_{\beta\alpha} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & A_{12} & \bar{A}_{16} & A_{16} & \bar{B}_{11} & B_{12} & \bar{B}_{16} & B_{16} \\ A_{12} & \hat{A}_{22} & A_{26} & \hat{A}_{26} & B_{12} & \hat{B}_{22} & B_{26} & \hat{B}_{26} \\ \bar{A}_{16} & A_{26} & \bar{A}_{66} & A_{66} & \bar{B}_{16} & B_{26} & \bar{B}_{66} & B_{66} \\ A_{16} & \hat{A}_{26} & A_{66} & \hat{A}_{66} & B_{16} & \hat{B}_{26} & B_{66} & \hat{B}_{66} \\ \bar{B}_{11} & B_{12} & \bar{B}_{16} & B_{16} & \bar{D}_{11} & D_{12} & \bar{D}_{16} & D_{16} \\ B_{12} & \hat{B}_{22} & B_{16} & \hat{B}_{16} & D_{12} & \hat{D}_{22} & D_{26} & \hat{D}_{26} \\ \bar{B}_{16} & B_{26} & \bar{B}_{66} & B_{66} & \bar{D}_{16} & D_{26} & \bar{D}_{66} & D_{66} \\ B_{16} & \hat{B}_{26} & B_{66} & \hat{B}_{66} & D_{16} & \hat{D}_{26} & D_{66} & \hat{D}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{0\alpha} \\ \varepsilon_{0\beta} \\ \varepsilon_{0\alpha\beta} \\ \varepsilon_{0\beta\alpha} \\ \kappa_\alpha \\ \kappa_\beta \\ \kappa_{\alpha\beta} \\ \kappa_{\beta\alpha} \end{bmatrix} \tag{13}$$

and

$$\begin{bmatrix} Q_\alpha \\ Q_\beta \\ P_\alpha \\ P_\beta \end{bmatrix} = \begin{bmatrix} \bar{A}_{55} & A_{45} & \bar{B}_{55} & B_{45} \\ A_{45} & \hat{A}_{44} & B_{45} & \hat{B}_{44} \\ \bar{B}_{55} & B_{45} & \bar{D}_{55} & D_{45} \\ B_{45} & \hat{B}_{44} & D_{45} & \hat{D}_{44} \end{bmatrix} \begin{bmatrix} \gamma_{0\alpha z} \\ \gamma_{0\beta z} \\ -\psi_\beta/R_{\alpha\beta} \\ -\psi_\alpha/R_{\alpha\beta} \end{bmatrix} \tag{14}$$

where

$$\left.\begin{aligned} A_{ij} &= \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(h_k - h_{k-1}) \\ B_{ij} &= \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(h_k^2 - h_{k-1}^2) \\ D_{ij} &= \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(h_k^3 - h_{k-1}^3) \end{aligned}\right\} \quad i,j = 1,2,6$$

$$\left.\begin{aligned} A_{ij} &= \sum_{k=1}^{N} K_i K_j \bar{Q}_{ij}^{(k)} (h_k - h_{k-1}) \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{N} K_i K_j \bar{Q}_{ij}^{(k)} (h_k^2 - h_{k-1}^2) \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{N} K_i K_j \bar{Q}_{ij}^{(k)} (h_k^3 - h_{k-1}^3) \end{aligned}\right\} \quad i,j = 4,5 \tag{15}$$

$$\left.\begin{aligned} \bar{A}_{ij} &= A_{ij\alpha} + \frac{B_{ij\alpha}}{R_\beta}, \quad \hat{A}_{ij} = A_{ij\beta} + \frac{B_{ij\beta}}{R_\alpha} \\ \bar{B}_{ij} &= B_{ij\alpha} + \frac{D_{ij\alpha}}{R_\beta}, \quad \hat{B}_{ij} = B_{ij\beta} + \frac{D_{ij\beta}}{R_\alpha} \\ \bar{D}_{ij} &= D_{ij\alpha} + \frac{E_{ij\alpha}}{R_\beta}, \quad \hat{D}_{ij} = D_{ij\beta} + \frac{E_{ij\beta}}{R_\alpha} \end{aligned}\right\} \quad i,j = 1,2,4,5,6$$

where K_i and K_j in the above equations are shear correction coefficients (Chang, 1993), typically taken at 5/6 and where

$$\left.\begin{aligned} A_{ijn} &= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \bar{Q}_{ij}^{(k)} \frac{\mathrm{d}z}{1+z/R_n} = R_n \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \ln\left(\frac{R_n + h_k}{R_n + h_{k-1}}\right) \\ B_{ijn} &= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \bar{Q}_{ij}^{(k)} \frac{z\,\mathrm{d}z}{1+z/R_n} = R_n \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \left[(h_k - h_{k-1}) - R_n \ln\left(\frac{R_n + h_k}{R_n + h_{k-1}}\right)\right] \\ D_{ijn} &= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \bar{Q}_{ij}^{(k)} \frac{z^2\,\mathrm{d}z}{1+z/R_n} \\ &= R_n \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \left[\frac{1}{2}\{(R_n+h_k)^2 - (R_n+h_{k-1})^2\} - 2R_n(h_k - h_{k-1}) - R_n^2 \ln\left(\frac{R_n + h_k}{R_n + h_{k-1}}\right)\right] \\ E_{ijn} &= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \bar{Q}_{ij}^{(k)} \frac{z^3\,\mathrm{d}z}{1+z/R_n} \\ &= R_n \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \left[\begin{aligned} &\frac{1}{3}\{(R_n+h_k)^3 - (R_n+h_{k-1})^3\} + \frac{3}{2} R_n \{(R_n+h_k)^2 - (R_n+h_{k-1})^2\} \\ &-3R_n^2(h_k - h_{k-1}) - R_n^3 \ln\left(\frac{R_n + h_k}{R_n + h_{k-1}}\right) \end{aligned}\right] \end{aligned}\right\} n = \alpha, \beta \tag{16}$$

4. Numerical verification

Tables 1 and 2 show a comparison between the accurate stress resultants obtained here with those valid for plates and commonly used for shells. Note that a and b are the side lengths of the

Table 1
Non-dimensional stiffness parameters for homogeneous deep isotropic shells ($E = E_1 = E_2, \nu = 0.3, a/b = 1, a/R_\beta = 2, a/R_{\alpha\beta} = 0, a/h = 10$)

	Plate approx.		Accurate equations		Plate approx.		Accurate equations		Plate approx.		Accurate equations	
(i, j)	$\frac{A_{ij}}{E_2 a^2}$	$\frac{\bar{A}_{ij}}{E_2 a^2}$	$\frac{\hat{A}_{ij}}{E_2 a^2}$	Error*	$\frac{100 B_{ij}}{E_2 a^3}$	$\frac{100 \bar{B}_{ij}}{E_2 a^3}$	$\frac{100 \hat{B}_{ij}}{E_2 a^3}$	Error*	$\frac{1000 D_{ij}}{E_2 a^4}$	$\frac{1000 \bar{D}_{ij}}{E_2 a^4}$	$\frac{1000 \hat{D}_{ij}}{E_2 a^4}$	Error*
Cylindrical shells ($R_\alpha/R_\beta = 0$)												
(1, 1)	0.109890	0.109890	NA	0.0	0	0.018306	NA	100	0.091575	0.091572	NA	0.0
(2, 2)	0.109890	NA	0.110257	0.3	0	NA	−0.018416	100	0.091757	NA	0.095128	0.6
(6, 6)	0.038460	0.038460	0.038588	0.3	0	0.006407	−0.006446	100	0.032050	0.032049	0.032244	0.6
Hyperbolic paraboloidal shells ($R_\alpha/R_\beta = -1$)												
(1, 1)	0.109890	0.110627	NA	0.6	0	0.036851	NA	100	0.091575	0.092685	NA	1.2
(2, 2)	0.109890	NA	0.110627	0.0	0	NA	−0.036851	100	0.091575	NA	0.092682	1.2
(6, 6)	0.038460	0.038718	0.038718	0.0	0	0.012987	−0.012897	100	0.032050	0.032437	0.032437	1.2

* E.g. error = $|(\bar{A}_{ij} - A_{ij}) \times 100/\bar{A}_{ij}|$.

Table 2
Non-dimensional stiffness parameters for [0, 90] laminated thick deep shells ($E_1/E_2 = 15, G_{12}/E_2 = 0.5, G_{13}/E_2 = 0.5, v_{12} = 0.3, a/b = 1, a/R_\beta = 2, a/R_{\alpha\beta} = 0, a/h = 10$)

	Plate approx.		Accurate equations		Plate approx.		Accurate equations		Plate approx.		Accurate equations	
(i, j)	$\frac{A_{ij}}{E_2 a^2}$	$\frac{\bar{A}_{ij}}{E_2 a^2}$	$\frac{\hat{A}_{ij}}{E_2 a^2}$	Error*	$\frac{100 B_{ij}}{E_2 a^3}$	$\frac{100 \bar{B}_{ij}}{E_2 a^3}$	$\frac{100 \hat{B}_{ij}}{E_2 a^3}$	Error*	$\frac{1000 D_{ij}}{E_2 a^4}$	$\frac{1000 \bar{D}_{ij}}{E_2 a^4}$	$\frac{1000 \hat{D}_{ij}}{E_2 a^4}$	Error*
Cylindrical shells ($R_\alpha/R_\beta = 0$)												
(1, 1)	0.804829	0.769634	NA	4.5	−1.760563	−1.626488	NA	8.2	0.670691	0.624178	NA	7.0
(2, 2)	0.804829	NA	0.772156	4.2	1.760563	NA	1.634540	7.7	0.670691	NA	0.630454	7.0
(6, 6)	0.050000	0.049999	0.050167	0.3	0	0.008329	−0.008379	100	0.041667	0.042892	0.041918	0.6
Hyperbolic paraboloidal shells ($R_\alpha/R_\beta = -1$)												
(1, 1)	0.804829	0.739450	NA	8.8	−1.760563	−1.508390	NA	16.7	0.670691	0.590178	NA	13.6
(2, 2)	0.804829	NA	0.739450	8.8	1.760563	NA	−1.508390	16.7	0.670691	NA	0.590178	13.6
(6, 6)	0.050000	0.050335	0.050335	0.6	0	0.016767	−0.016767	100	0.041667	0.042170	0.042170	1.2

* E.g. error $= |(\bar{A}_{ij} - A_{ij}) \times 100/\bar{A}_{ij}|$.

shell (Fig. 1), and h is the total thickness of the shell (Fig. 2). In Table 1, isotropic thick ($a/h = 10$) shells of deep curvature ($a/R_\beta = 2$) are used. The A_{ij} terms are the membrane stress resultants and the B_{ij} terms present the stretching–bending coupling. The D_{ij} terms are the bending stress resultants. All the stress resultant terms have been non-dimensionalized as shown in the tables. Spherical shells are not considered here because stress resultants for spherical shells are the same as those for plates. This has been verified numerically. One important observation is that the use of plate stress resultants did not predict an extension–bending coupling for isotropic material, common for laminated composites, while the accurate stress resultants did predict such coupling. Such observation was earlier made for curved beams (Qatu, 1993). Other than the above phenomenon, the accurate equations do not seem to improve the accuracy of the stress resultants by more than 2%. Further numerical results for a thickness ratio of five yielded an improvement of no more than 5%. Hence, the accurate equations may not be needed for isotropic shells and the term $(1+z/R)$ in the denominator can be neglected yielding stress resultants similar to those of plates. The expansion of such a term by use of a geometric series is not needed and yields undesired complexities as in Vlasov's equations (Leissa, 1973).

Laminated composite thick and deep shells are considered in Table 2. As shown there, the use of the plate stress resultant equations did yield up to 16% error in the stress resultant coefficients. Some researchers used the plate stress resultant equations for shells with a thickness ratio of five. This 16% error is more than what is typically reported as an error due to ignoring shear deformation. Such use will lead to greater errors. It is noticed that such an error is more for hyperbolic paraboloidal shells than that for cylindrical shells.

Table 3 shows the stiffness parameters for laminated composite thick shallow shells. The curvature is taken at the limit of shallow shell theory ($a/R = 0.5$). As can be seen from the table, the plate approximate equations lead to a maximum of 4% error for hyperbolic paraboloidal shells. Thus, the accurate equations may not be necessary for laminated shallow shells.

Table 4 shows results obtained using parameters usually considered as the limits of thin shell theories. As noted, using the plate equations yield a maximum error of approximately 4%. Thus, the accurate equations need not be used for thin laminated and isotropic shells.

In conclusion, ignoring the $(1+z/R)$ term in the denominator yields considerable inaccuracy for laminated composite deep thick shells. The accurate stress resultant equations derived here should then be used for these shells.

5. Energy functionals

The strain energy of the body when under elastic deformation can be written as:

$$U = \frac{1}{2}\int_V (\sigma_\alpha \varepsilon_\alpha + \sigma_\beta \varepsilon_\beta + \sigma_z \varepsilon_z + \sigma_{\alpha\beta}\gamma_{\alpha\beta} + \sigma_{\alpha z}\gamma_{\alpha z} + \sigma_{\beta z}\gamma_{\beta z})\,\mathrm{d}V \tag{17}$$

where V is volume and

$$\mathrm{d}V = \mathrm{d}S_\alpha \quad \mathrm{d}S_\beta \quad \mathrm{d}z = AB(1+z/R_\alpha)(1+z/R_\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}z \tag{18}$$

Substitute (5), (9) and (10) into the above equations, the strain energy functional can then be written as:

Table 3
Non-dimensional stiffness parameters for [0, 90] laminated thick shallow shells ($E_1/E_2 = 15, G_{12}/E_2 = 0.5, G_{13}/E_2 = 0.5, v_{12} = 0.3, a/b = 1, a/R_\beta = 0.5, a/R_{\alpha\beta} = 0, a/h = 10$)

(i,j)	Plate approx.		Accurate equations		Plate approx.		Accurate equations		Plate approx.		Accurate equations	
	$\frac{A_{ij}}{E_2a^2}$	$\frac{\bar{A}_{ij}}{E_2a^2}$	$\frac{\hat{A}_{ij}}{E_2a^2}$	Error*	$\frac{100B_{ij}}{E_2a^3}$	$\frac{100\bar{B}_{ij}}{E_2a^3}$	$\frac{100\hat{B}_{ij}}{E_2a^3}$	Error*	$\frac{1000D_{ij}}{E_2a^4}$	$\frac{1000\bar{D}_{ij}}{E_2a^4}$	$\frac{1000\hat{D}_{ij}}{E_2a^4}$	Error*
Cylindrical shells ($R_\alpha/R_\beta = 0$)												
(1, 1)	0.804829	0.769043	NA	1.1	−1.760563	−1.727094	NA	1.9	0.670691	0.659080	NA	1.8
(2, 2)	0.804829	NA	0.796208	1.1	1.760563	NA	1.727633	1.9	0.670691	NA	0.659956	1.6
(6, 6)	0.050000	0.050000	0.050000	0.0	0	0.002079	−0.002080	100	0.041667	0.041973	0.041682	0.6
Hyberpolic paraboloidal shells ($R_\alpha/R_\beta = -1$)												
(1, 1)	0.804829	0.787553	NA	2.2	−1.760563	−1.694570	NA	3.9	0.670690	0.649178	NA	3.3
(2, 2)	0.804829	NA	0.787553	2.2	1.760563	NA	1.694570	3.9	0.670690	NA	0.649178	3.3
(6, 6)	0.050000	0.050021	0.050021	0.0	0	0.004168	−0.004168	100	0.041666	0.041698	0.041698	0.1

* E.g. error $= |(\bar{A}_{ij} - A_{ij}) \times 100/\bar{A}_{ij}|$.

Table 4
Non-dimensional stiffness parameters for [0, 90] laminated thin deep shells ($E_1/E_2 = 15, G_{12}/E_2 = 0.5, G_{13}/E_2 = 0.5, \nu_{12} = 0.3, a/b = 1, a/R_\beta = 2, a/R_{\alpha\beta} = 0, a/h = 40$)

	Plate approx.		Accurate equations		Plate approx.		Accurate equations		Plate approx.		Accurate equations	
(i,j)	$\frac{A_{ij}}{E_2a^2}$	$\frac{\bar{A}_{ij}}{E_2a^2}$	$\frac{\hat{A}_{ij}}{E_2a^2}$	Error*	$\frac{100B_{ij}}{E_2a^3}$	$\frac{100\bar{B}_{ij}}{E_2a^3}$	$\frac{100\hat{B}_{ij}}{E_2a^3}$	Error*	$\frac{1000D_{ij}}{E_2a^4}$	$\frac{1000\bar{D}_{ij}}{E_2a^4}$	$\frac{1000\hat{D}_{ij}}{E_2a^4}$	Error*
Cylindrical shells ($R_\alpha/R_\beta = 0$)												
(1, 1)	0.201207	0.199008	NA	1.1	−0.110035	−0.107940	NA	1.9	0.010480	0.010308	NA	1.9
(2, 2)	0.201207	NA	0.199049	1.1	1.110035	NA	0.107939	1.9	0.010480	NA	0.010312	1.6
(6, 6)	0.012500	0.012500	0.012503	0.0	0	0.000130	−0.000130	100	0.000651	0.000651	0.000651	0.0
Hyberpolic paraboloidal shells ($R_\alpha/R_\beta = -1$)												
(1, 1)	0.201207	0.196888	NA	2.2	−0.110035	−0.105911	NA	3.9	0.010480	0.010143	NA	3.3
(2, 2)	0.201207	NA	0.196888	2.2	0.110035	NA	0.105911	3.9	0.010480	NA	0.010143	3.3
(6, 6)	0.012500	0.012505	0.012505	0.0	0	0.000261	−0.000261	100	0.000651	0.000652	0.000652	0.1

* E.g. error = $|(\bar{A}_{ij} - A_{ij}) \times 100/\bar{A}_{ij}|$.

$$U = \frac{1}{2}\int_\alpha \int_\beta \Bigg(N_\alpha \varepsilon_{0\alpha} + N_\beta \varepsilon_{0\beta} + N_{\alpha\beta}\varepsilon_{0\alpha\beta} + N_{\beta\alpha}\varepsilon_{0\beta\alpha} + Q_\alpha \gamma_{0\alpha z} + Q_\beta \gamma_{0\beta z} + M_\alpha \kappa_\alpha + M_\beta \kappa_\beta + M_{\alpha\beta}\kappa_{\alpha\beta} + M_{\beta\alpha}\kappa_{\beta\alpha} - \frac{P_\alpha \psi_\beta}{R_{\alpha\beta}} - \frac{P_\beta \psi_\alpha}{R_{\alpha\beta}} \Bigg) AB\,\mathrm{d}\alpha\,\mathrm{d}\beta \quad (19)$$

which can be written in terms of displacements by substituting eqns (6) and (7) in the above equation.

External work can be done on a shell element by applying distributed forces and moments as well as end reaction. The total work can be written as:

$$W = \int_\alpha \int_\beta (q_\alpha u_0 + q_\beta v_0 + q_n w_0 + m_\alpha \psi_\alpha + m_\beta \psi_\beta) AB\,\mathrm{d}\alpha\,\mathrm{d}\beta + \int_\beta (N_{0\alpha} u_0 + N_{0\alpha\beta} v_0 + Q_{0\alpha} w_0 + M_{0\alpha}\psi_\alpha + M_{0\beta\alpha}\psi_\beta) B\,\mathrm{d}\beta + \int_\alpha (N_{0\beta} v_0 + N_{0\beta\alpha} u_0 + Q_{0\beta} w_0 + M_{0\beta}\psi_\beta + M_{0\alpha\beta}\psi_\alpha) A\,\mathrm{d}\alpha \quad (20)$$

where $N_{0\alpha}$, $N_{0\beta}$, $N_{0\alpha\beta}$ and $N_{0\beta\alpha}$ are in-plane normal and shear force results; $Q_{0\alpha}$ and $Q_{0\beta}$ are out of plane shear force resultants; $M_{0\alpha}$ and $M_{0\beta}$ are bending moments and $M_{0\alpha\beta}$ and $M_{0\beta\alpha}$ are twisting moments. Work done by external concentrated forces can be added to the energy term by multiplying the force by the displacement component in its direction, or by expanding the force in a series.

Kinetic energy can be given by:

$$T = \frac{\rho}{2}\int_V \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right) \mathrm{d}V \quad (21)$$

where ρ is mass density per unit volume and V is volume.

Substituting eqns (4) into the kinetic energy expression and integrating with respect to z, yields:

$$T = \frac{1}{2}\int_A \left(\left(\left(\frac{\partial u_0}{\partial t}\right)^2 + \left(\frac{\partial v_0}{\partial t}\right)^2 + \left(\frac{\partial w_0}{\partial t}\right)^2 \right) \bar{I}_1 + \left(\left(\frac{\partial u_0}{\partial t}\right)\left(\frac{\partial \psi_\alpha}{\partial t}\right) + \left(\frac{\partial v_0}{\partial t}\right)\left(\frac{\partial \psi_\beta}{\partial t}\right) \right) \bar{I}_2 + \left(\left(\frac{\partial \psi_\alpha}{\partial t}\right)^2 + \left(\frac{\partial \psi_\beta}{\partial t}\right)^2 \right) \bar{I}_3 \,\mathrm{d}\alpha\,\mathrm{d}\beta \quad (22)$$

where:

$$\bar{I}_1 = \left(I_1 + I_2 \left(\frac{1}{R_\alpha} + \frac{1}{R_\beta} \right) + \frac{I_3}{R_\alpha R_\beta} \right)$$

$$\bar{I}_2 = \left(I_2 + I_3 \left(\frac{1}{R_\alpha} + \frac{1}{R_\beta} \right) + \frac{I_4}{R_\alpha R_\beta} \right)$$

$$\bar{I}_3 = \left(I_3 + I_4\left(\frac{1}{R_\alpha} + \frac{1}{R_\beta}\right) + \frac{I_5}{R_\alpha R_\beta}\right) \tag{23}$$

$$[I_1, I_2, I_3, I_4, I_5] = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \rho^{(k)}[1, z, z^2, z^3, z^4]\,\mathrm{d}z \tag{24}$$

It should be mentioned that the terms I_2 and I_4 are non-zero only if the material densities are not symmetric about the middle surface. These quantities are zeros for unsymmetrical lamination if the same material is used.

6. Equations of motion

In order to develop a consistent set of equations, the equations of motion will be derived using Hamilton's principle:

$$\delta \int_{t_0}^{t_1} (U - W - T)\,\mathrm{d}t = 0 \tag{25}$$

Substituting the equations for potential energy, external work and kinetic energy and performing the integration by parts and then setting the coefficients of the displacement variations equal to zero, in a normal manner, yields the following equations of motion:

$$\frac{\partial}{\partial\alpha}(BN_\alpha) + \frac{\partial}{\partial\beta}(AN_{\beta\alpha}) + \frac{\partial A}{\partial\beta}N_{\alpha\beta} - \frac{\partial B}{\partial\alpha}N_\beta + \frac{AB}{R_\alpha}Q_\alpha + \frac{AB}{R_{\alpha\beta}}Q_\beta - ABq_\alpha = AB(\bar{I}_1\ddot{u}_0^2 + \bar{I}_2\ddot{\psi}_\alpha^2)$$

$$\frac{\partial}{\partial\beta}(AN_\beta) + \frac{\partial}{\partial\alpha}(AN_{\alpha\beta}) + \frac{\partial B}{\partial\alpha}N_{\beta\alpha} - \frac{\partial A}{\partial\beta}N_\alpha + \frac{AB}{R_\beta}Q_\beta + \frac{AB}{R_{\alpha\beta}}Q_\alpha + ABq_\beta = AB(\bar{I}_1\ddot{v}_0^2 + \bar{I}_2\ddot{\psi}_\beta^2)$$

$$-AB\left(\frac{N_\alpha}{R_\alpha} + \frac{N_\beta}{R_\beta} + \frac{N_{\alpha\beta} + N_{\beta\alpha}}{R_{\alpha\beta}}\right) + \frac{\partial}{\partial\alpha}(BQ_\alpha) + \frac{\partial}{\partial\beta}(AQ_\beta) + ABq_n = AB(\bar{I}_1\ddot{w}_0^2)$$

$$\frac{\partial}{\partial\alpha}(BM_\alpha) + \frac{\partial}{\partial\beta}(AM_{\beta\alpha}) + \frac{\partial A}{\partial\beta}M_{\alpha\beta} - \frac{\partial B}{\partial\alpha}M_\beta - ABQ_\alpha + \frac{AB}{R_{\alpha\beta}}P_\beta + ABm_\alpha = AB(\bar{I}_2\ddot{u}_0^2 + \bar{I}_3\ddot{\psi}_\alpha^2)$$

$$\frac{\partial}{\partial\beta}(AM_\beta) + \frac{\partial}{\partial\alpha}(BM_{\alpha\beta}) + \frac{\partial B}{\partial\alpha}M_{\beta\alpha} - \frac{\partial A}{\partial\beta}M_\alpha - ABQ_\beta + \frac{AB}{R_{\alpha\beta}}P_\alpha + ABm_\beta = AB(\bar{I}_2\ddot{v}_0^2 + \bar{I}_3\ddot{\psi}_\beta^2) \tag{26}$$

where the two dots over the terms present the second derivative of these terms with respect to time.

7. Boundary conditions

Hamilton's principle will also yield boundary terms that are consistent with the other equations. These boundary terms for the boundaries with α = constant are:

$$N_{0\alpha} - N_\alpha = 0 \qquad \text{or} \qquad u_0 = 0$$

$$N_{0\alpha\beta}-N_{\alpha\beta}=0 \quad \text{or} \quad v_0=0$$
$$Q_{0\alpha}-Q_\alpha=0 \quad \text{or} \quad w_0=0$$
$$M_{0\alpha}-M_\alpha=0 \quad \text{or} \quad \psi_\alpha=0$$
$$M_{0\alpha\beta}-M_{\alpha\beta}=0 \quad \text{or} \quad \psi_\beta=0 \tag{27}$$

Similarly for β = constant. Chang (1993) obtained the same equations.

8. Doubly curved shells

Consider a shell with the following characteristics:

(1) Constant radii of curvature R_α and R_β and a radius of twist of infinity, i.e. $(1/R_{\alpha\beta})=0$.
(2) Constant Lamé parameters. They equal one unit.

This is shown to be the case for shallow shells, twisted plates, cylindrical shells and barrel shells. Constant Lamé parameters cannot be applied to general shells.

Substituting eqns (6), (7), (13) and (14) into eqns (26) yields the equilibrium equations in terms of displacements. These equations are proven useful when exact solutions are desired. The equations can be written as:

$$L_{ij}u_i+M_{ij}\ddot{u}_i=q_i \tag{28}$$

where

$$u_i=[u_0,v_0,w_0,\psi_\alpha,\psi_\beta]^T$$

and

$$q_i=[-q_\alpha,-q_\beta,-q_n,-m_\alpha,-m_\alpha]^T$$

$$L_{11}=\bar{A}_{11}\frac{\partial^2}{\partial\alpha^2}+2A_{16}\frac{\partial^2}{\partial\alpha\,\partial\beta}+\hat{A}_{66}\frac{\partial^2}{\partial\beta^2}-\frac{\bar{A}_{55}}{R_\alpha^2}$$

$$L_{12}=\bar{A}_{16}\frac{\partial^2}{\partial\alpha^2}+(A_{12}+A_{66})\frac{\partial^2}{\partial\alpha\,\partial\beta}+\hat{A}_{26}\frac{\partial^2}{\partial\beta^2}-\frac{A_{45}}{R_\alpha R_\beta}$$

$$L_{13}=\left[\frac{\bar{A}_{11}+\bar{A}_{55}}{R_\alpha}+\frac{A_{12}}{R_\beta}\right]\frac{\partial}{\partial\alpha}+\left[\frac{A_{16}+A_{45}}{R_\alpha}+\frac{\hat{A}_{26}}{R_\beta}\right]\frac{\partial}{\partial\beta}$$

$$L_{14}=\bar{B}_{11}\frac{\partial^2}{\partial\alpha^2}+2B_{16}\frac{\partial^2}{\partial\alpha\,\partial\beta}+\hat{B}_{66}\frac{\partial^2}{\partial\beta^2}+\frac{\bar{A}_{55}}{R_\alpha}$$

$$L_{15}=\bar{B}_{16}\frac{\partial^2}{\partial\alpha^2}+(B_{12}+B_{66})\frac{\partial^2}{\partial\alpha\,\partial\beta}+\hat{B}_{26}\frac{\partial^2}{\partial\beta^2}-\frac{\hat{A}_{45}}{R_\alpha}$$

The stiffness parameters L_{ij} in eqn (28) are:

$$L_{22} = \bar{A}_{66}\frac{\partial^2}{\partial\alpha^2} + 2A_{26}\frac{\partial^2}{\partial\alpha\,\partial\beta} + \hat{A}_{22}\frac{\partial^2}{\partial\beta^2} - \frac{\hat{A}_{44}}{R_\beta^2}$$

$$L_{23} = \left[\frac{A_{26}+A_{45}}{R_\beta} + \frac{\bar{A}_{16}}{R_\alpha}\right]\frac{\partial}{\partial\alpha} + \left[\frac{\hat{A}_{22}+\hat{A}_{44}}{R_\beta} + \frac{A_{12}}{R_\alpha}\right]\frac{\partial}{\partial\beta}$$

$$L_{24} = \bar{B}_{16}\frac{\partial^2}{\partial\alpha^2} + (B_{12}+B_{66})\frac{\partial^2}{\partial\alpha\,\partial\beta} + \hat{B}_{26}\frac{\partial^2}{\partial\beta^2} + \frac{A_{45}}{R_\beta}$$

$$L_{25} = \bar{B}_{66}\frac{\partial^2}{\partial\alpha^2} + 2B_{26}\frac{\partial^2}{\partial\alpha\,\partial\beta} + \hat{B}_{22}\frac{\partial^2}{\partial\beta^2} + \frac{\hat{A}_{44}}{R_\beta}$$

$$L_{33} = -\bar{A}_{55}\frac{\partial^2}{\partial\alpha^2} - 2A_{45}\frac{\partial^2}{\partial\alpha\,\partial\beta} - \hat{A}_{44}\frac{\partial^2}{\partial\beta^2} + \frac{\bar{A}_{11}}{R_\alpha^2} + \frac{2A_{12}}{R_\alpha R_\beta} + \frac{\hat{A}_{22}}{R_\beta^2}$$

$$L_{34} = \left[-\bar{A}_{55} + \frac{\bar{B}_{11}}{R_\alpha} + \frac{B_{12}}{R_\beta}\right]\frac{\partial}{\partial\alpha} + \left[-A_{45} + \frac{B_{16}}{R_\alpha} + \frac{\hat{B}_{26}}{R_\beta}\right]\frac{\partial}{\partial\beta}$$

$$L_{35} = \left[-A_{45} + \frac{\bar{B}_{16}}{R_\alpha} + \frac{B_{26}}{R_\beta}\right]\frac{\partial}{\partial\alpha} + \left[-\hat{A}_{44} + \frac{B_{12}}{R_\alpha} + \frac{\hat{B}_{22}}{R_\beta}\right]\frac{\partial}{\partial\beta}$$

$$L_{44} = -\bar{A}_{55} + \bar{D}_{11}\frac{\partial^2}{\partial\alpha^2} + 2\bar{D}_{16}\frac{\partial^2}{\partial\alpha\,\partial\beta} + \hat{D}_{66}\frac{\partial^2}{\partial\beta^2}$$

$$L_{45} = -A_{45} + \bar{D}_{16}\frac{\partial^2}{\partial\alpha^2} + (D_{12}+D_{66})\frac{\partial^2}{\partial\alpha\,\partial\beta} + \hat{D}_{26}\frac{\partial^2}{\partial\beta^2}$$

$$L_{55} = -\hat{A}_{44} + \bar{D}_{66}\frac{\partial^2}{\partial\alpha^2} + 2D_{26}\frac{\partial^2}{\partial\alpha\,\partial\beta} + \hat{D}_{22}\frac{\partial^2}{\partial\beta^2} \quad (29)$$

The mass parameters in eqn (28) are:

$$M_{11} = M_{22} = M_{33} = -\bar{I}_1$$

$$M_{14} = M_{25} = -\bar{I}_2$$

$$M_{44} = M_{55} = -\bar{I}_3$$

$$\text{all other} \quad M_{ij} = 0 \quad (30)$$

9. Exact solutions for simple support boundaries

Equations (28) are valid for forced vibration problems. They can be specialized to static analysis by letting the time derivative equal zero. To obtain the free vibration problem, the static loading (q_i) is set to zero.

Assume that the shell is constructed of cross-ply laminates. Thus,

$$A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0 \tag{31}$$

For shear diaphragm boundaries (S2) (Leissa, 1973), the well-known Navier solution can be applied to obtain an exact solution. The displacement and shear functions are assumed to be:

$$u_0(\alpha,\beta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}\cos(a_m\alpha)\sin(b_n\beta)\sin(\omega t)$$

$$v_0(\alpha,\beta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn}\sin(a_m\alpha)\cos(b_n\beta)\sin(\omega t)$$

$$w_0(\alpha,\beta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\sin(a_m\alpha)\sin(b_n\beta)\sin(\omega t)$$

$$\psi_\alpha(\alpha,\beta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \psi_{\alpha mn}\cos(a_m\alpha)\sin(b_n\beta)\sin(\omega t)$$

$$\psi_\beta(\alpha,\beta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \psi_{\beta mn}\sin(a_m\alpha)\cos(b_n\beta)\sin(\omega t) \tag{32}$$

where $a_m = m\pi/a$ and $b_n = n\pi/b$.

Substituting eqns (32) into eqns (28) and using the Fourier expansion for the loading functions yields

$$[K]\{\Delta\}+\omega^2[M]\{\Delta\} = -\{F\} \tag{33}$$

where $[K]$, $[M]$ are the stiffness and mass matrices, respectively, ω is the frequency, $\{F\}$ is the forcing function and

$$\{\Delta\} = \{U_{mn}, V_{mn}, W_{mn}, \psi_{\alpha mn}, \psi_{\beta mn}\}^T \tag{34}$$

The above equations can be used directly for the natural frequencies.

10. Comparison with published results

The following natural frequency parameter is used for the subsequent analysis:

$$\Omega = \omega a^2\sqrt{\rho/E_2h^2} \tag{35}$$

Tables 5 and 6 show comparisons of the natural frequency parameters with previously published results for cylindrical and spherical open shells, respectively. FSD theories were used by Librescu et al. (1989) and Bhimaraddi (1991). Librescu et al. and Bhimaraddi also used Higher-order Shear Deformation (HSD) theories. Bhimaraddi used the three-dimensional (3-D) theory of elasticity to obtain exact analytical solutions. The previously described Navier solution was used to obtain these 3-D results.

In HSD theories, boundary conditions at the upper and lower surface of the shell can be satisfied and there is no need for shear correction factors. Despite this, the results shown indicate that HSD theories do not always yield better results than FSD ones. The results show that all FSD and HSD

Table 5
Non-dimensional frequency parameters Ω for [0, 90] cylindrical shells ($E_1/E = 25, G_{12}/E_2 = 0.5, G_{13}/E_2 = 0.5, G_{23}/E_2 = 0.2, \nu_{12} = 0.3, k^2 = 5/6, a/b = 1, a/h = 10$)

R/a	Present theory	Equation of Appendix A*		Bhimaraddi (1991)		Librescu (1989)	
			3-D	FSD	HSD	FSD	HSD
1	10.643	10.667	10.4085	10.7475	10.9189		
2	9.4428	9.4577	9.3627	9.3653	9.5664		
3	9.1755	9.1860	9.1442	9.0563	9.2642		
4	9.0731	9.0811	9.0613	8.9403	9.1506		
5	9.0221	9.0286	9.0200	8.8840	9.0953	8.931	8.959
10	8.9446	8.9479	8.9564	8.8026	9.0150	8.897	8.933
20	8.9194	8.9199	8.9341	8.7779	8.9904	8.894	8.934
∞	8.9001	8.9001	8.9179	8.7640	8.9761	8.900	8.944

* Same as Chang (1993).

Table 6
Non-dimensional frequency parameters Ω for [0, 90] spherical shells ($E_1/E = 25, G_{12}/E_2 = 0.5, G_{13}/E_2 = 0.5, G_{23}/E_2 = 0.2, \nu_{12} = 0.3, k^2 = 5/6, a/b = 1, a/h = 10$)

R/a	Present theory	Equation of Appendix A*		Bhimaraddi (1991)		Librescu (1989)	
			3-D	FSD	HSD	FSD	HSD
1	14.4746	14.4746	13.9974	14.8008	14.9075		
2	10.7478	10.7478	10.5528	10.8054	10.9708		
3	9.7822	9.7822	9.6917	9.7455	9.9330		
4	9.4102	9.4102	9.3637	9.3332	9.5306		
5	9.2309	9.2309	9.2065	9.1338	9.3361	9.247	9.292
10	8.9843	8.9843	8.9912	8.8584	9.0679	8.989	9.033
20	8.9213	8.9213	8.9363	8.7877	8.9992	8.922	8.966
∞	8.9001	8.9001	8.9179	8.7640	8.9761	8.900	8.944

* Same as present theory, agrees with Chang (1993) to the fourth or fifth decimal.

theories deviate from the 3-D theory of elasticity as the shell becomes deeper. This indicates that the error in these theories may very well be due to the fact that the term $1+z/R$ was ignored. Previous FSD and HSD theories further show that the deviation from 3-D is higher for spherical shells than cylindrical ones.

The present theory shows closer approximation to the 3-D results when compared with the FSD theories of Librescu et al. and Bhimaraddi. The present theory also shows better approximation than HSD theories.

It should be mentioned that the equations used here (13)–(16) do suffer from numerical instability when one or both radii of curvature become large. The expansion used in the Appendix yielded results that are close to those obtained here and offers numerical stability when the radii values become large, like in shallow shells. Such equations should be used for shallower shells.

11. Conclusions

A complete and consistent set of equations are derived for laminated composite deep thick shells. These equations include an initial pre-twist, which most other theories ignored. These equations further include accurate force and moment resultant equations by including the $(1+z/R)$ terms (in the denominator of the stress resultant integrands) in the integration, which almost all other theories ignored. Numerical results verified that, for the force and moment resultants as well as the natural frequencies, these terms should be included for deep thick composite shells. Including these terms in free vibration analysis yielded frequencies that are close to those obtained by 3-D theory of elasticity. Considerations of these terms yielded significant enhanced accuracy for laminated composite thick curved beams (Qatu, 1993).

It has been concluded that using the plate approximation equations for stiffness parameters of isotropic thick shells leads to an error of 2% and for thin laminated shells or thick shallow shells leads to an error of 4%. Thus, the plate approximation equations may be used for these theories without a major sacrifice in the accuracy. For deep thick shells, however, the accurate equations presented here [eqns (13)–(16)] or their geometric expansion alternative (equations provided in the Appendix) should be used. It has been demonstrated that the accurate equations yield closer approximation than previously published FSD and HSD theories for the natural frequencies when compared with the three-dimensional theory of elasticity. This has been demonstrated by obtaining results of various theories for spherical and cylindrical shells. Hyperbolic paraboloidal shells should yield similar results and will be investigated in a later study.

Appendix: Alternative derivation of the stress resultant equations

The term $(1+z/R_\alpha)/(1+z/R_\beta)$ shown in various locations in eqns (11) and (12) for force and moment resultants is expanded here using a geometric series, thus,

$$\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k}\frac{\left(1+\dfrac{z}{R_{\alpha k}}\right)}{\left(1+\dfrac{z}{R_{\beta k}}\right)}\mathrm{d}z=\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k}\left(1+\frac{z}{R_{\alpha k}}\right)\left(1-\frac{z}{R_{\beta k}}+\frac{z^2}{R_{\beta k}^2}-\cdots\right)\mathrm{d}z \tag{A1}$$

where R_α and R_β are the radii of curvature of the midsurface of the shell, $R_{\alpha k}$ and $R_{\beta k}$ are the radii of curvature of the kth layer, taken at the midsurface of that layer.

Multiplying the integrand of the right-hand side of eqn (A1) and neglecting higher-order terms, yields

$$\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k}\frac{\left(1+\dfrac{z}{R_{\alpha k}}\right)}{\left(1+\dfrac{z}{R_{\beta k}}\right)}\mathrm{d}z \approx \sum_{k=1}^{N}\int_{h_{k-1}}^{h_k}\left(1+z\left(\frac{1}{R_\alpha}-\frac{1}{R_\beta}\right)\right)\mathrm{d}z \tag{A2}$$

Using the above equations and letting

$$c_0 = \left(\frac{1}{R_\alpha}-\frac{1}{R_\beta}\right) \tag{A3}$$

will yield the following equations

$$\left.\begin{aligned}\bar{A}_{ij} &= A_{ij}-c_0B_{ij}, & \hat{A}_{ij} &= A_{ij}+c_0B_{ij}\\ \bar{B}_{ij} &= B_{ij}-c_0D_{ij}, & \hat{B}_{ij} &= B_{ij}+c_0D_{ij}\\ \bar{D}_{ij} &= D_{ij}-c_0E_{ij}, & \hat{D}_{ij} &= D_{ij}+c_0E_{ij}\end{aligned}\right\} \quad i,j=1,2,4,5,6 \tag{A4}$$

where

$$\left.\begin{aligned}A_{ij} &= \sum_{k=1}^{N}\bar{Q}_{ij}^{(k)}(h_k-h_{k-1})\\ B_{ij} &= \frac{1}{2}\sum_{k=1}^{N}\bar{Q}_{ij}^{(k)}(h_k^2-h_{k-1}^2)\\ D_{ij} &= \frac{1}{3}\sum_{k=1}^{N}\bar{Q}_{ij}^{(k)}(h_k^3-h_{k-1}^3)\\ E_{ij} &= \frac{1}{4}\sum_{k=1}^{N}\bar{Q}_{ij}^{(k)}(h_k^4-h_{k-1}^4)\end{aligned}\right\} \quad i,j=1,2,6 \tag{A5}$$

and

$$\left.\begin{aligned}A_{ij} &= \sum_{k=1}^{N}K_iK_j\bar{Q}_{ij}^{(k)}(h_k-h_{k-1})\\ B_{ij} &= \frac{1}{2}\sum_{k=1}^{N}K_iK_j\bar{Q}_{ij}^{(k)}(h_k^2-h_{k-1}^2)\\ D_{ij} &= \frac{1}{3}\sum_{k=1}^{N}K_iK_j\bar{Q}_{ij}^{(k)}(h_k^3-h_{k-1}^3)\end{aligned}\right\} \quad i,j=4,5 \tag{A6}$$

The above equations are similar to those obtained by Chang (1993) and Leissa and Chang (1996).

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